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On almost arcwise connected and almost Peano continua

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1. Introduction. The concept of almost arcwise connectedness was introduced by K.R.Kellum[8] in 1976 as a kind of connectivity preserved by an almost continuous retraction. In 1980 he characterized the almost continuous image of a Peano continuum in the class of second countable T_1-spaces, and obtained the notion of "almost Peano"[10]. Recently Hagopian [5] established some precise results on almost arcwise connected continua without a dense arc component.

In this note we shall make a survey of almost arcwise connected and almost Peano continua and some open questions.

2. Almost continuous functions and almost arcwise connectedness.

DEFINITION 1.[8] A topological space X is said to be <u>almost arcwise connected</u> if each two non-empty open sets of X can be joined by an arc in X.

Each of the following implications is obvious, and non-reversibly:

$$\begin{matrix}\text{arcwise}\\ \text{connected}\end{matrix} \longrightarrow \begin{matrix}\text{almost arcwise}\\ \text{connected}\end{matrix} \longrightarrow \text{connected}$$

DEFINITION 2.[11] A function f from a topological space X to another topological space Y is <u>almost</u> <u>continuous</u> if for any open set G of $X \times Y$ containing the graph $\Gamma(f)$, there is a continuous function g from X to Y whose graph $\Gamma(g)$ lies in G.

EXAMPLE 1. Let f be a function from $I = [0,1]$ to itself defined by

$$f(x) = \begin{cases} \sin 1/x & : \quad 0 < x \leqq 1 \\ 0 & : \quad x = 0. \end{cases}$$

Then, f is almost continuous.

THEOREM 1.[8] A Hausdorff almost continuous image of an almost arcwise connected T_1-space is also almost arcwise connected.

EXAMPLE 2.[8] In the square J^2, $J = [-1,1]$, let

$$A = \{(x, \sin 1/x) : 0 < x \leqq 1\} \cup \{(0,y) : -1 \leqq y \leqq 1\}.$$

Then the set A is an almost continuous retract of J^2; that is, there is an almost continuous function $r : J^2 \to A$ such that $r|A = 1_A$.

Let $B = A \cup \{(x,0): -1 \leqq x \leqq 0\}$. The set B is not an almost continuous retract of J^2, because B is not almost arcwise connected.

3. Almost Peano continua.

DEFINITION 3.[10] A topological space X is said to be <u>almost Peano</u> if for each finite collection $\{U_1,\ldots,U_n\}$ of non-empty open sets of X there is a path $\sigma : I \to X$ such that $\sigma(I) \cap U_i \neq \emptyset$ for each i.

A <u>continuum</u> is a nondegenerate compact connected metric space. A <u>Peano continuum</u> is a locally connected continuum. A continuum X is <u>unicoherent</u> if for each pair A, B of its subcontinua such that $X = A \cup B$, the intersection $A \cap B$ is connected. A continuum is <u>decomposable</u> if it is the union of two proper subcontiua; otherwise it is <u>indecomposable</u>. A continuum is <u>hereditarily</u> P (P a topological property) if each subcontinuum has the property P. A continuum is a <u>λ-dendroid</u> if it is hereditarily unicoherent and hereditarily decomposable.

Kellum gave a characterization of the almost continuous image of a Peano continuum:

THEOREM 2.[10] Let X be a second countable T_1-space. Then X is an almost continuous image of a Peano continuum if and only if X is almost Peano.

In the class of continua we have the following implications:

$$\begin{array}{c}\text{a continuum with}\\ \text{a dense arc component}\end{array} \longrightarrow \text{almost Peano} \longrightarrow \begin{array}{c}\text{almost arcwise}\\ \text{connected}\end{array}$$

Kellum[10] thought that the converse of the first arrow was true, but C.L.Hagopian gave a counterexample.

EXAMPLE 3.[5] There exists an indecomposable hereditarily unicoherent almost plane continuum that does not have a dense arc component.

This example is constructed by modification of the Knaster's indecomposable continuum with one endpoint.

The converse of the second arrow is also erroneous:

EXAMPLE 4.[10] There exists an almost arcwise connected continuum which is not almost Peano.

We have the Knaster's indecomposable continuum K with two endpoints and a triangle T. Replace each side of T with a copy of K where the endpoints of K go to vertices. Then the resulting continuum Y satisfies the requirement we need.

THEOREM 3.[5] Let X be an almost arcwise connected λ-dendroid. If (1) X is planar,

or (2) X has only countably many arc components,

then X has one dense arc component and every other arc component of X is nowhere dense.

The hypothesis of "almost arcwise connected" in this theorem cannot be replaced by "almost Peano" without assumptions (1) or (2) [5;Example 3].

From this theorem we have the following:

COROLLARY [5] (1) An almost arcwise connected hereditarily unicoherent plane continuum without a dense arc component has uncountably many arc components.

(2) An almost arcwise connected λ-dendroid without arc component has uncountably many arc components.

Each of the proofs is done by the method of contraposition.

The first part of above corollary is not true, when we replace the assumption of "almost arcwise connected hereditarily unicoherent and plane" by "almost Peano". We have such an example in R^3 [5;Example 4].

4. Open questions.

QUESTION 1.[6] If X is an arcwise connected acyclic continuum in the square J^2, is it an almost continuous retract of J^2?

QUESTION 2.[8] If Y is an almost continuous retract of J^2 and Z is an almost continuous retract of Y, then is Z an almost continuous retract of J^2?

The answer to this question is "yes", if Y and Z are compact.

QUESTION 3.[7] Is there an indecomposable continuum which is an almost continuous retract of J^2?

QUESTION 4.[10] Can the second countability hypothesis in Theorem 2 be weakened?

QUESTION 5.[5] Does every almost Peano plane continuum without a dense arc component have uncountably many components?

5. NOTES. The study of non-continuous functions was initiated by O.H.Hamilton[4] in 1957. He generalized the Brouwer fixed point theorem to certain non-continuous function, called a connectivity function. J.Stallings[11] observed a gap in Hamilton's proof. He remedied this defect and defined the almost continuous functions to extend the Hamilton's theorem onto a compact polyhedron. K.Borsuk[11;Question 10] pointed out the hope of new fixed point theorem using non-continuous retracts.

Thus, much of the study of this area is concerned with the fixed point theory.

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